WALSH – HADAMARD TRANSFORMATION OF A CONVOLUTION

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Abstract

A convolution is mathematical operation used in signal processing, in the homomorphous signal processing and digital image processing (e.g. image interpolation). In regard of computational complexity of the convolution in the time domain, it used to calculate in the other domain. Exp. $x(n) * h(n) \to X(\Omega) \cdot H(\Omega)$, resp. $X(\Omega) \cdot H(\Omega)$, shows that a convolution in the time domain corresponds to multiplication in the Z domain, respectively frequency domain. This paper shows utilization of Walsh-Hadamard orthogonal transformations for convolution.

Keywords

Convolution, Walsh-Hadamard transformation, dyadic convolution

1. Introduction

It is well known that for many orthogonal transformations, the image of the convolution of a pair of functions is equal to the product of their transforms. In order to obtain an analogous rule for Walsh–Hadamard transformation, the convolution should be replaced by the dyadic convolution.

Recall that if

$$m = \sum_{i=0}^{\infty} m_i 2^i \qquad n = \sum_{i=0}^{\infty} n_i 2^i .$$

are binary expansions of non-negative integers m, n then their dyadic sum is

$$m \oplus n = \sum_{i=0}^{\infty} |m_i - n_i| \cdot 2^i .$$

$R^{m\times n}$ denotes the space of all $m\times n$ matrices with real entries.

DEFINITION 1. Let $N_k = 2^k$, $k = 0, 1, 2\ldots$ A (column) vector $z = x * y \in R^{Nk\times 1}$ is called a dyadic convolution of $x, y \in R^{Nk\times 1}$ if its n-th entry is

$$z(n) = \frac{1}{N_k} \sum_{d=0}^{N_k-1} x(d)\, y(n \oplus d), \qquad n = 0,1,\ldots N_k - 1.$$

$$x \in R^{N_{k+1}\times 1}$$

can be identified with a pair $x_0, x_1 \in R^{Nk\times 1}$:

$$x = \begin{bmatrix} x_0^{\mathrm{T}} & x_1^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{1}$$

where

$$x_0 = [x(0), x(1), \ldots, x(N_k - 1)]^{\mathrm{T}},$$
$$x_1 = [x(N_k), x(N_k + 1), \ldots, x(N_{k+1} - 1)]^{\mathrm{T}}.$$

Using this identification we obtain for the convolution

$$z = x * y, \quad x, y \in R^{N_{k+1}\times 1}$$

$$z(n) = \frac{1}{2^{k+1}} \sum_{d=0}^{2^{k+1}-1} x(d)\, y(n \oplus d) \tag{2}$$

$$= \frac{1}{2}\left\{ \frac{1}{2^k} \sum_{d=0}^{2^k-1} x(d)\, y(n \oplus d) + \frac{1}{2^k} \sum_{d=2^k}^{2^{k+1}-1} x(d)\, y(n \oplus d) \right\}.$$

If $0 \le n < 2^k$ (i.e. for coordinates of z_0), we obtain in the first sum of (2) $n \oplus d < 2^k$, and in the second sum $n \oplus d =$ $= 2^k + [n \oplus (d - 2^k)]$, consequently

$$z_0 = \frac{1}{2}[x_0 * y_0 + x_1 * y_1].$$

Similarly one can see that $z_1 = 0.5\, [x_0 * y_1 + x_1 * y_0]$, i.e.

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} * \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_0 * y_0 + x_1 * y_1 \\ x_0 * y_1 + x_1 * y_0 \end{bmatrix}. \tag{3}$$

Recall that Hadamard matrices are defined inductively by

$$H_1 = [1],\ H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \ldots, H_{2^{k+1}} = \begin{bmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{bmatrix} \tag{4}$$

and the Walsh-Hadamard transformation $x \in R^{2^k\times 1}$ is

$$X_h = \frac{1}{2^k} H_{2^k} x . \tag{5}$$

THEOREM 1. Let k be a non-negative integer, $N_k = 2^k$ and $x, y \in R^{Nk \times 1}$. Then

$$z = x * y \Rightarrow Z_h(n) = X_h(n)Y_h(n), \quad n = 0, 1, \ldots 2^k\text{-}1,$$

i.e. Z_h is coordinate wise multiplication of X_h and Y_h.

PROOF. The theorem can be proved by mathematical induction:

1. The assertion is obvious for $k = 0$.
2. Suppose that the theorem holds for $k \geq 0$. We prove that it holds also for k+1. Using the relations (3) – (5) we obtain for arbitrary

$$x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, \quad y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}, \quad x_0, x_1, y_0, y_1 \in R^{N_k \times 1}$$

$$X_h = \frac{1}{2^{k+1}} H_{2^{k+1}} x = \frac{1}{2^{k+1}} \begin{bmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

and a similar relation holds for y. Consequently

$$X_h = \frac{1}{2} \begin{bmatrix} X_{0h} + X_{1h} \\ X_{0h} - X_{1h} \end{bmatrix} \text{ and } Y_h = \frac{1}{2} \begin{bmatrix} Y_{0h} + Y_{1h} \\ Y_{0h} - Y_{1h} \end{bmatrix}. \quad (6)$$

Using (6) and the assumption that the theorem holds for k, we obtain for

$$z = x * y = \frac{1}{2} \begin{bmatrix} x_0 * y_0 + x_1 * y_1 \\ x_0 * y_1 + x_1 * y_0 \end{bmatrix}$$

the following:

$$Z_h = \frac{1}{2^{k+1}} H_{2^{k+1}} z$$

$$= \frac{1}{4} \frac{1}{2^k} \begin{bmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{bmatrix} \begin{bmatrix} x_0 * y_0 + x_1 * y_1 \\ x_0 * y_1 + x_1 * y_0 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} X_{0h}Y_{0h} + X_{1h}Y_{1h} + X_{0h}Y_{1h} + X_{1h}Y_{0h} \\ X_{0h}Y_{0h} + X_{1h}Y_{1h} - X_{0h}Y_{1h} - X_{1h}Y_{0h} \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} (X_{0h} + X_{1h})(Y_{0h} + Y_{1h}) \\ (X_{0h} - X_{1h})(Y_{0h} - Y_{1h}) \end{bmatrix} = X_h Y_h. \quad (7)$$

This finishes the proof.

REMARK. Theorem 1 holds for another order of Walsh functions since the corresponding transform of $x \in R^{Nk \times 1}$ can be obtain from its Hadamard transform X_h by a fixed permutation of components of X_h.

2. Interpolation Using Walsh (or Sequence) Ordered WHT

For the evaluation of the interpolation of the $N1 \times N2$ image I, we can even use the $(\text{WHT})_\text{w}$, where we use the following routine that also applies dyadic convolution of the image and filter in $(\text{WHT})_\text{w}$ domain:

1. In the spatial domain, the image I, that we want to interpolate, is alternately added with zeros to get image I', i.e.

$$I = \begin{bmatrix} I_{0,0} & I_{0,1} & I_{0,2} & \cdots & I_{0,N-1} \\ I_{1,0} & I_{1,1} & I_{1,2} & \cdots & I_{1,N-1} \\ I_{2,0} & I_{2,1} & I_{2,2} & \cdots & I_{2,N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{N-1,0} & I_{N-1,1} & I_{N-1,2} & \cdots & I_{N-1,N-1} \end{bmatrix} \Rightarrow \quad (8)$$

$$I' = \begin{bmatrix} I_{0,0} & 0 & I_{0,1} & 0 & \cdots & I_{0,N-1} & 0 \\ 0 & I_{1,0} & 0 & I_{1,1} & \cdots & 0 & I_{1,N-1} \\ I_{2,0} & 0 & I_{2,1} & 0 & \cdots & I_{2,N-1} & 0 \\ 0 & I_{3,0} & 0 & I_{3,1} & \cdots & 0 & I_{3,N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{N-2,0} & 0 & I_{N-2,1} & 0 & \cdots & I_{N-2,N-1} & 0 \\ 0 & I_{N-1,0} & 0 & I_{N-1,1} & \cdots & 0 & I_{N-1,N-1} \end{bmatrix}.$$

2. We compute the $(\text{WHT})_\text{w}$ spectrum of the image I' – $W_{I'}(k1, k2)$.
3. We use the filter type 3 or 4 – their transfer functions are defined for $N = 8$ as follows:

Type 3:

$$H_3(k1, k2) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (9)$$

Type 4: (10)

$$H_4(k1, k2) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0.5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0.5 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0.5 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0.5 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0.5 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e., multiply element by element $(\mathrm{WHT})_\mathrm{w}$'s of I' with transfer function of the filter

$$Y(k1,k2) = W_{\mathrm{I'}}(k1,k2) \cdot H(k1,k2). \tag{11}$$

4. The reconstructed image is given by the equation

$$y(n1,n2) = 2 \cdot (U_{\mathrm{N}1})^{\mathrm{T}} \cdot Y(k1,k2) \cdot U_{N2} \tag{12}$$

where U_{N1} is the base of 1-D $(\mathrm{WHT})_\mathrm{w}$.

Fig. 1 Image LENA 256×256 a) original, b) under sampled, 1. order and alternately added with zeros, c) interpolated 2D-DFT, d) interpolated $(\mathrm{WHT})_\mathrm{w}$+ filter 3, e) interpolated $(\mathrm{WHT})_\mathrm{w}$+ filter 4

3. Conclusion

For comparison, I mention the computational complexity of using the fast Fourier transform and using Walsh – Hadamard transform. Computational complexity for a N fast Fourier transform coefficients $F_x(k)$, $k = 0, 1\ldots N\text{-}1$ is $N \cdot \log_2 N$ complex multiplies and adds. On the other side, N^2 additions and subtractions are required to compute the $(\mathrm{WHT})_\mathrm{w}$ coefficients $W_x(k)$, where $k = 0, 1\ldots N\text{-}1$. A fast algorithm $(\mathrm{FWHT})_\mathrm{w}$ yields the $W_x(k)$, $k = 0, 1\ldots N\text{-}1$, in $N \cdot \log_2 N$ additions and subtractions. In frequency domain (FFT), we required complex multiplications element by element, in comparison with $(\mathrm{WHT})_\mathrm{w}$, where we do only real multiplications element by element.

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